Mexican contributions to Noncommutative Theories

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Abstract. In this paper we summarize the Mexican contributions to the subject of Noncommutative theories. These contributions span several areas: Quantum Groups, Noncommutative Field Theories, Hopf algebra of renormalization, Deformation Quantization, Noncommutative Gravity, and Noncommutative Quantum Mechanics.

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1. INTRODUCTION

One of the most important open problems in theoretical physics is to understand the microscopic structure of the space-time. Because of the interplay of Quantum Mechanics and General Relativity at the Planck length scale, there are good reasons for believing that at short distances the structure of space-time may no longer be considered as a 4-dimensional continuum. Furthermore, under these premises the very concept of manifold as an underlying mathematical structure in the construction of unified physical theories, applicable to distances of the order of the Planck length, becomes questionable and some people have been convinced that a new paradigm of geometric space is needed that would allow us to incorporate into our theoretical formalisms small-scale structures completely different from those to which we are usually accustomed. Among mathematicians mainly, one such outstanding paradigm is the noncommutative geometry invented by Connes, which considers a new calculus, the so called spectral calculus, based on operators in Hilbert space and the use of the tools of spectral analysis [1]. This geometry has among its features that it includes ordinary Riemannian space; discrete spaces are treated on the same footing as the continuum, thus allowing for a mixture of the two. Furthermore, it allows the possibility of noncommuting coordinates.

On the other hand, in Physics it would appear reasonable the need to extend the phase-space noncommutativity of quantum mechanics to a noncommutativity of space-time in order to study the microstructure of the space-time. It then becomes a major issue the study of the noncommutative spaces. These spaces are characterized by the commutation relations satisfied by their coordinate operators,

$$[\hat{x}^i, \hat{x}^j] = i\hbar\Theta^{ij}. \quad (1)$$
Here the constant parameters of the non-commutativity are given by the real and antisymmetric matrix elements $\hbar \Theta^{ij}$, having dimensions of area. Noncommutativity of space-time has been in the literature for many years [2] and was studied in some systems. However, recently this subject has received a renewed interest, and now there are several reasons why in physics we are interested in this subject. For example, the quantum Hall effect, one of the most studied phenomena in condensed matter, presents noncommutativity in the canonical coordinates and momenta [3]. Further evidence along this line of thought has been provided by recent developments in string theory [4]. In their proposal, Seiberg and Witten have found signals of noncommutativity in the description of the low energy excitations of open strings (possibly attached to D-branes) in the presence of a Neveu-Schwarz constant background $B$-field. In the decoupling limit, the low energy effective field theory is described in terms of a supersymmetric gauge theory. By means of two different regularization schemes, two versions of the same effective theory are obtained in this limit, corresponding to the commutative and the noncommutative gauge theories. The independence on the regularization scheme leads to an equivalence of these theories which gives rise to the so-called Seiberg-Witten map. This map relates the noncommutative and the commutative gauge theories, both of them with the same number of physical degrees of freedom. It has also been shown recently that in noncommutative field theories the Seiberg-Witten map can be interpreted as a field dependent gravitational background [5]. In fact, it is not difficult to show that a similar interpretation can be carried out even at the level of quantum mechanics on noncommutative phase-space. These recent results, as well as others (c.f. examples of noncommutative geometry in field theory listed in [6]), have generated a considerable interest to understand the role played by noncommutative geometry in different theoretical sectors of physics.

One way to deal with these noncommutative spaces is to construct a new kind of field theory, changing the standard product of the fields by the star product (Weyl-Moyal):

$$ (f \ast g)(x) = \exp\left(\frac{i}{2} \Theta^{ij} \partial_i \partial_j \right) f(x) g(y)|_{x=y}, \quad (2) $$

where $f$ and $g$ are infinitely differentiable functions. In this theory some interesting results have been found [7]; for instance, it was shown that there is a relation between the infrared and ultraviolet divergences [8]. Several interactions have been analyzed in this context, for example in the case of the Standard Model see the references [9] and [10].

If, on the other hand, one assumes the commutation rules:

$$ [\hat{x}^i, \hat{x}^j] = i\hbar \Theta^{ij}, \quad [\hat{x}^i, \hat{p}_j] = i\hbar \delta^i_j, \quad [\hat{p}_i, \hat{p}_j] = 0. \quad (3) $$

a noncommutative quantum mechanics can be formulated, of which some relevant results have already been obtained [11], [12].

Although none of the above mentioned apparently conceptually different approaches and their variants are anywhere close to a final theory of grand unification, and probably no single one of this directions will succeed in producing it, there appears to be emerging a common denominator of noncommutativity in some of their ingredients which points to the fact that when considering the problem of coordinates below the Planck
length, there is no good reason to presume that the texture of space-time will still have a 4-dimensional continuum structure.

The purpose of this paper is to review the Mexican contributions to the subject of noncommutative theories. The subject includes contributions no only from physicists, and mathematical physicists, also there are important contributions from mathematicians, and we have tried to incorporate most of the references including topics related to particles and fields. We divide these contributions in several areas that we order chronologically. In Section 2 we review the contributions to Quantum Groups and Braid Groups. In Section 3 we present the contributions to Deformation Quantization. In Section 4 we present the Mexican contributions to Noncommutative Field theories, Noncommutative Gravity and Noncommutative Quantum Cosmology. In Section 5 we summarize the area of Noncommutative Geometry and Hopf algebras of renormalization. Finally, in Sec. 6 we present the contributions to Noncommutative Quantum Mechanics.

2. QUANTUM GROUPS

Quantum groups were first introduced independently by Drinfeld [13], Jimbo [14], and Woronowicz [15]. The motivation that leads to the study of these mathematical structures was the consideration of the symmetries of the Yang-Baxter equations, in the case of Drinfeld and Jimbo. On the other hand, starting from a deformation of the universal enveloping algebra of $SU(2)$ Woronowicz got the same structure. From the mathematical point of view Quantum groups are not groups, are Quasitriangular Hopf algebras [16]. But in some sense, these structures represent to Noncommutative Geometry the equivalent of a Lie Group to Differential Geometry. In this area of study there were very important contributions to the subject by Mexican researchers. The main contribution is the book Quantum Groups in two-dimensional Physics by M. Ruiz-Altaba, et al [17]. This book was the first serious treatment of the subject by physicists and is a fundamental reference for this area. In the case of papers, an interesting contribution to the area is [18]. In this article the authors found an explicit quantum group structure of WZNW models, in their Coulomb gas representation. They construct the quantum group generators in the Chevalley basis as operators creating, destroying or counting screening charges, and show that they satisfy all the required properties. In another paper [19], the same authors describe in detail the contour representation of finite quantum groups which appears naturally in the Coulomb gas representation of conformal field theories. Another important contribution to this area is the construction of Quantum Clifford algebras from spinor representations [20]. Here using the quantum group formalism of bi-covariant bimodules it was possible to built the quantum analogues of Dirac and Laplace operators and to construct quantum Spin(n) groups [21]. This construction was one of the first fully consistent Quantum Clifford algebras. One interesting extension of this subject was the construction of Involutive braided Spin groups [22]. These mathematical structures are generalizations of quantum groups, that include a braid operator. Other contributions to this area are [23], [24], [25] and [26].
3. DEFORMATION QUANTIZATION

The goal of classical and quantum mechanics is to study the evolution of a system. In classical mechanics the set of possible states of a system are points in the phase space and the observables are functions of a commutative algebra. In quantum mechanics the physical states form a projective Hilbert space and the set of observables are self-adjoint operators that form a noncommutative algebra. The aim of Deformation Quantization is to obtain the quantum theory of a system by deforming the product that is used to multiply the observables of the theory.

In mathematics, the deformation of an object is a family of objects depending on a parameter. In the case of Deformation Quantization it is possible to define a family of associative products, star products, that depend on the deformation parameter $\hbar$ (or $\theta$). In this context, research in Mexico has produced several interesting proposals. For example in [27] was shown how the reduced Self-dual Yang-Mills theory described by the Nahm equations can be carried over to the Weyl-Wigner-Moyal formalism employed in Self-dual gravity. This shows the existence of a correspondence between BPS magnetic monopoles and space-time hyper-Kähler metrics. On the other hand, the Deformation quantization of the bosonic string was constructed in [28]. In this paper it was shown that the light-cone gauge is the most convenient classical description to perform the quantization of bosonic strings in the deformation quantization formalism. The mass spectrum, propagators and the Virasoro algebra were finally described within this deformation quantization scheme. This work was extended in [29]. Here, the deformation quantization of scalar and abelian gauge classical free fields was studied and the Stratonovich-Weyl quantizer, together with star-products and Wigner functionals were obtained. Another interesting contributions to this area are: [30] and [31]. For Kahler manifolds it is known that an alternative procedure to quantize these spaces is that given by Berezin in Ref. [32]. In this procedure it is also defined an star product called the Berezin star. This was used in geometric quantum mechanics in [33]. It is known that several field theories have as moduli spaces which are compact Kahler manifolds. This idea was precisely implemented in [34] for the study of a quantization of Chern-Simons gauge theories and in [35] for WZW and coset models.

The more general associative star product is due to Kontsevich [36]. This construction allows us to define star products on general phase spaces, with coordinate dependent and degenerated symplectic structures. In physics this construction has some applications, for example in [37] it was shown that the Hopf algebra of renormalization and the universal formula of Kontsevich for quantum deformation are equivalent. This result could be useful to study in more depth the perturbative structure of quantum field theory.

4. NONCOMMUTATIVE FIELD THEORIES

Noncommutative gravity has been formulated in the literature by using different approaches (for instance, see, Refs. [38, 39, 40]). More recently Chamseddine has made several proposals for noncommutative formulations of Einstein’s gravity [41, 42, 43], where a Moyal deformation is done. A more recent proposal of a noncommutative deformation of Einstein-Hilbert Lagrangian in four dimensions is given in [44]. The study
of models of noncommutative gravity, trying to imitate Yang-Mills gauge theory, arising from string or M-theory, is very important. Such models could be obtained starting from those formulations of gravitation which are based on a gauge principle. Two of these formulations are the topological gravity and the Plebański formulation of self-dual gravity \[45\], from which the hamiltonian Ashtekar’s formulation \[46\] can be obtained.

In México, different proposals of noncommutative gravity in four dimensions have been formulated recently. A proposal for gauge topological gravity is given in terms of a noncommutative gauge invariant action \[47\]. Here the possibility of noncommutative topological gravity arising in the same manner as a Yang-Mills theory is explored. The Seiberg-Witten map is used to construct such a theory based on a SL(2, C) complex connection, from which the Euler characteristic and the signature invariant are obtained. This gives a way towards the description of noncommutative gravitational instantons as well as noncommutative local gravitational anomalies.

Moreover, using a noncommutative formulation of Plebański’s self-dual gravity a noncommutative theory of pure Einstein theory in four dimensions was obtained \[48\]. In order to do that, the Seiberg-Witten map was used. It is shown that the noncommutative torsion constraint is solved by the vanishing of commutative torsion. Also, the noncommutative corrections to the action are computed up to second order. In the case of linearized gravity an interesting approach was presented in \[49\].

On the other hand a formulation of noncommutative quantum cosmology starting from a noncommutative parametrization of the minisuperspace has been also given \[50\]. Here it was proposed a model for noncommutative quantum cosmology by means of a deformation of minisuperspace. For the Kantowski-Sachs metric the exact wave function was found. Thus, wave packets are constructed and it is shown that new quantum states appear that “compete” to be the most probable state, in clear contrast with the commutative case. A tunnelling process could be possible among these states. In this same direction, the CFT construction of S-branes describing the rolling and bouncing tachyons is analyzed in the context of a \(\theta\)-noncommutative deformation of minisuperspace. Half s-brane and s-brane in the noncommutative minisuperspace are studied and exact analytic solutions, involving the noncommutative parameter \(\theta\) and compatible with the boundary conditions at infinity, are found. Also it was performed a comparison with the usual commutative minisuperspace. This was described in Ref. \[51\]. Another different proposal to noncommutative quantum cosmology is given in \[52\].

Going back to topic of noncommutative gravity, a noncommutative description of topological half-flat gravity in four dimensions was formulated. BRST symmetry of this topological gravity is deformed through a twisting of the usual BRST quantization of noncommutative gauge theories. It is argued also that the resulting moduli space of instantons is characterized by the solutions of a noncommutative version of the Plebański’s heavenly equation \[53\].

The analogous calculation in Yang-Mills theories can also be addressed, for instance the cohomological Yang-Mills theory is formulated on a noncommutative differentiable four manifold through the \(\theta\)-deformation of its corresponding BRST algebra. The resulting noncommutative field theory is a natural setting to define the \(\theta\)-deformation of Donaldson invariants and they are interpreted as a mapping between the Chevalley-Eilenberg homology of noncommutative spacetime and the Chevalley-Eilenberg coho-
mology of noncommutative moduli of instantons. In the process one can find that in the weak coupling limit the quantum theory is localized at the moduli space of noncommutative instantons [54].

Gravitational axial and chiral anomalies in a noncommutative space are examined through the explicit perturbative computation of one-loop diagrams in various dimensions [55]. The analysis depend on how gravity is coupled to noncommutative matter fields. The Delbourgo-Salam computation of the gravitational axial anomaly contribution to the pion decay into two photons, is studied in detail in this context. In the process one can see that the two-dimensional chiral pure gravitational anomaly does not receive noncommutative corrections. Pure gravitational chiral anomaly in 4\(k+2\) dimensions with matter fields being chiral fermions of spin-1/2 and spin-3/2, is discussed and a noncommutative correction is found. In this paper, the mixed anomalies are considered in both cases. Finally in [56], the influence of higher dimensions in noncommutative field theories was studied. For this purpose, the bosonic sector of a recently proposed 6 dimensional SU(3) orbifold model for the electroweak interactions was analyzed. The corresponding noncommutative theory is constructed by means of the Seiberg-Witten map in 6D. In the reduced bosonic interactions for a 4D theory, new couplings (with respect to those known in others 4D noncommutative formulations of the Standard Model) were found using the Seiberg-Witten map.

5. NONCOMMUTATIVE GEOMETRY AND HOPF ALGEBRAS OF RENORMALIZATION

In mathematics, Gaussian and Riemannian geometric spaces are usually defined as manifolds where the metric is given by the geodesic distance

\[ d_\gamma(x,y) = \inf \gamma \{ \text{length of paths } \gamma \text{ from } x \text{ to } y \}. \]  

(4)

However, in order to reduce the geometry to algebraic form and to arrive at a formulation which can be extended to noncommutative spaces, Connes [1] has proposed, as starting point, the following dual form of (4):

\[ d(x,y) = \sup \{ |f(x) - f(y)|, f \in \mathcal{A}, \| df \| \}, \]  

(5)

where \( \mathcal{A} \) is the algebra of \( C^\infty(M) \) functions over \( M \), and \( ds \) is the line element in Riemannian geometry. To measure distances in a possible noncommutative space \( X \), equation (5) is generalized by first introducing a Fredholm module \((\mathcal{H}, F)\) over the involutive algebra \( \mathcal{A} \). Here \( F \) is a selfadjoint involutive operator acting on the Hilbert space \( \mathcal{H} = L^2(M, S) \) of square integrable sections of the irreducible spinor bundle over \( M \). The differential calculus is quantized by using the operator quantum theoretic notion for the differential

\[ df = [F, f], \]  

(6)

where \( f \in \mathcal{A} \). One further specifies a metric structure on \( X \) by defining a unit length via an operator of the form

\[ G = (dx^\mu)^* g_{\mu\nu} (dx^\nu), \]  

(7)
with $x^\mu \in \mathcal{A}$, then $dx = [F, x]$, and $g_{\mu\nu}$ is a positive element of the matrix algebra $M_q(\mathcal{A})$. Thus, $G$ is a positive compact operator, and we can think of its positive square root as the line element of Riemannian geometry, $\sqrt{G} = ds$. Connes identifies this operator with the inverse of the Dirac operator. A more interesting idea is to think instead that the geometry of the space-time is dictated by Quantum Field Theory. According to Connes and Kreimer [57], this idea stresses the fact that space-time ought to be regarded as a derived concept. A remarkable result that gives support to this idea is the equivalence between the Hopf algebra of Connes-Moscovici [58], found in the context of Noncommutative Geometry, and the Hopf algebra of Kreimer [59], found in the context of Quantum Field Theory. Using this identification it is possible to associate the renormalization process in quantum field theory with a general mathematical procedure of extraction of finite values based on the Riemann-Hilbert problem [60].

In this area there were some interesting Mexican contributions. For example, the introduction of normal coordinates on the infinite dimensional group $G$ studied by Connes and Kreimer in their analysis of the Hopf algebra of rooted trees was proposed in [61]. Furthermore, the primitive elements of the algebra were studied and it was shown that they are generated by a simple application of the inverse Poincaré lemma, starting from a closed left invariant 1-form on $G$. For the special case of the ladder primitives, a second description was found that relates them to the Hopf algebra of functionals on a power series with the usual product. Either approach shows that the ladder primitives are given by the Schur polynomials. This analysis reduces considerably the renormalization procedure of primitive ladder Feynman diagrams. On the other hand in [62], the combinatorics resulting from the perturbative expansion of the transition amplitude in quantum field theories was analyzed, and its relation to the Hausdorff series. It was shown that in the context of these structures the power sum symmetric functionals of the perturbative expansion are Hopf primitives and that they are given by linear combinations of Hall polynomials, or diagrammatically by Hall trees. Furthermore, it was shown that each Hall tree corresponds to sums of Feynman diagrams each with the same number of vertices, external legs and loops. In addition, since the Lie subalgebra admits a derivation endomorphism, it was also shown that, with respect to it, these primitives are cyclic vectors generated by the free propagator, and thus provide a recursion relation by means of which the $(n+1)$-vertex connected Green functions can be derived systematically from the n-vertex ones. Another contributions to this area are: [63], [64], [65], [66], [67], [68].

6. NONCOMMUTATIVE QUANTUM MECHANICS

From the intrinsically noncommutative operator point of view, the development of a formulation for noncommutative quantum mechanics requires: (1) a specification of a representation for the phase-space algebra, (2) a specification of the Hamiltonian which governs the time evolution of the system and (3) a specification of the Hilbert space on which these operators and the other observables of the theory act. Regarding the choice of a representation for the intrinsic Heisenberg noncommutative phase-space algebra, several recent works in the literature have suggested to use a quantum mechanical equivalent to the Seiberg-Witten map [4], whereby the noncommutative Heisenberg
algebra is mapped into a commutative one [69], [70], [71]. Since in all generality this map admits many possible realizations, one could have in principle also many possible resulting self-consistent quantum mechanics of which the proper one could only be discerned by experiment. Moreover, since single particle quantum mechanics can be seen, in the free field or weak coupling limit, as a mini-superspace sector of quantum field theory where most degrees of freedom have been frozen. It is interesting to continue a more detailed study of exactly solvable models in noncommutative quantum mechanics and perhaps this results will be helpful both for the understanding of the effects of noncommutativity in field theory, as well as of its possible phenomenological consequences.

In this area, an interesting work produced in México is [72], where it was shown that corrections to the Newton’s second law appear if it is assumed that the phase space has a symplectic structure consistent with the rules of commutation of noncommutative quantum mechanics. In the central field case was found that the correction term breaks the rotational symmetry. In particular, for the Kepler problem, this term takes the form of a Coriolis force produced by the weak gravitational field far from a rotating massive object. Following this line of thinking the Kepler problem was studied in more depth in [73] and it was shown that a noncommutative parameter of the order of $10^{-58}m^2$ gives observable corrections to the movement of the solar system. In this way, modifications in the physics at smaller scales imply modifications at large scales, something similar to the UV/IR mixing. Another interesting contribution is [74], where the authors formulate non-relativistic classical and quantum mechanics in the noncommutative two dimensional plane. The approach used is based on the Galilei group, where the noncommutativity is seen as a central extension upon identification of the boost generators with the position operator. Furthermore, they perform a systematic study of the free particle, defined by the symmetries of the space-time, which include the noncommutativity. The symmetries at the classical level are analyzed in terms of Noether’s theorem. Canonical quantization is presented and the representation of the corresponding Heisenberg algebra is obtained. The path integral representation and Wigner distribution function in phase space are also discussed. Finally, they use Einstein’s model for a solid to corroborate that, according with intuition, the entropy is a growing function of $\theta$ in the low temperature regime, as a consequence of the space fuzziness. On the other hand in [75], following the idea that noncommutative quantum mechanics is related to noncommutative field theory, it was shown that introducing an extended Heisenberg algebra in the context of the Weyl-Wigner-Groenewold-Moyal formalism leads to a deformed product of the classical dynamical variables that is inherited at the level of quantum field theory. This allows to relate the operator space noncommutativity in quantum mechanics to the quantum group inspired algebra deformation noncommutativity in field theory.

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