\textbf{ψψ condensate in constant magnetic fields}

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We solve the Dirac equation in the presence of a constant magnetic field in (3 + 1) and (2 + 1) dimensions. Quantizing the fermion field, we calculate the \( ψψ \) condensate from first principles for parity conserving and violating Lagrangians for arbitrary field strength. We make a comparison with the results already known in the literature for some particular cases and point out the relevance of our work for possible physical applications.

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I. INTRODUCTION

Quantum electrodynamics (QED) exhibits special features in the presence of external magnetic fields. For example, fermions acquire masses dynamically in the presence of a constant magnetic field of arbitrary strength for any value of the electromagnetic coupling. This infrared phenomenon, established as universal in (3 + 1) and (2 + 1) dimensions, is dominated by the lowest Landau level (LLL) dynamics of electrons, and has been dubbed as magnetic catalysis [1–4]. QED4 in magnetic fields has relevance in the early universe or astrophysical environments where physics is likely to be influenced by the background magnetic field. On the other hand, QED3 has important applications in condensed matter physics, e.g., in high temperature superconductors [5], quantum Hall effect, and more recently, graphene [6]. In this connection, not only parity conserving but also parity violating models play a conspicuous role [7,8].

In this report, we solve the Dirac equation exactly in the presence of a homogeneous external magnetic field of arbitrary strength for parity conserving and violating fermionic Lagrangians. Expanding out the solutions in their Fourier modes, we calculate the \( ψψ \) condensate both in (3 + 1) dimensions as well as for the generalized case in (2 + 1) dimensions including parity violating mass terms. In the LLL approximation for (3 + 1) dimensions, our result reduces to that of [9] derived through Schwinger proper time method. In (2 + 1) dimensions, we consider both parity conserving and parity violating Lagrangians. It is well known that for the parity conserving case, the flavor U(2) breaks down to U(1) \( × U(1) \) spontaneously, even though fermions do not acquire mass [9,10]. In other words, they obtain a finite condensate when \( m → 0 \). This result is in sharp contrast with what is observed in (3 + 1) dimensions, [9]. Here the condensate \( \approx m \ln m → 0 \) as \( m → 0 \). Therefore, the existence of a finite \( ψψ \) condensate for massless bare fermions has been regarded as a specific (2 + 1)-dimensional phenomenon. In this work, we show that for parity nonconserving Lagrangian, (2 + 1) dimensions do not restrict themselves to this peculiarity. In the presence of both parity conserving and parity violating masses, \( m \) and \( m_p \) respectively, one can define parity even and parity odd condensates. The persistence of these condensates in the corresponding massless limit indicates that parity and chiral symmetry get spontaneously violated through interactions with the magnetic field. We can define convenient linear combinations of these condensates which separate the sectors of different fermion species. We denote them as \( \langle ψψ \rangle_+ \) and \( \langle ψψ \rangle_- \). We find that \( \langle ψψ \rangle_+ → 0 \) as mass \( m → 0 \). Thus, its behavior is similar to the condensate in (3 + 1) dimensions. On the other hand, \( \langle ψψ \rangle_- \) is finite even when mass \( m → 0 \).

We have organized this brief report as follows: In Sec. II, we solve Dirac equation for all (3 + 1)- and (2 + 1)-dimensional cases. We evaluate corresponding \( ψψ \) condensates in Sec. III. We then present our conclusions and discussions.

II. DIRAC EQUATION IN MAGNETIC FIELD

The Dirac equation in an external magnetic field is

\[
(iγ^μ \partial_μ + eγ^μ A_μ - m)ψ = 0.
\]

We first find its solutions in (3 + 1) dimensions. As we want to make a direct connection with (2 + 1) dimensions, we adopt the following representation of \( γ^μ \)

\[
γ^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad γ^i = i\hat{σ}^i = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},
\]

where \( \hat{σ} \) represents matrices \( σ^1 \) and \( σ^2 \). Similarly, \( \hat{σ} \) represents Pauli matrices \( σ_1 \) and \( σ_2 \). The fact that the first three matrices \( γ^i \), i.e., \( γ^0, γ^1, \) and \( γ^2 \) are in the block diagonal form will help make a comparison between (3 + 1)- and (2 + 1)-dimensional cases. We choose the magnetic field in the \( z \) direction. Working in the Landau gauge, we select \( A_μ = (0, 0, xB, 0) \). We can write the positive energy solutions as

\[
ψ_+ = \begin{pmatrix} ψ_+^1 \\ ψ_+^2 \end{pmatrix} = \begin{pmatrix} ψ_0^1 \\ -iB ψ_0^0 \end{pmatrix}, \quad ψ_- = \begin{pmatrix} ψ_-^1 \\ ψ_-^2 \end{pmatrix} = \begin{pmatrix} iB ψ_−^2 \\ ψ_−^0 \end{pmatrix}.
\]
where observations: normalization factor. We can readily make the following polynomials, \( p \), direction, and the negative energy solutions as

\[
\begin{align*}
\psi^1_p &= N_p e^{-i(E_p|t-p)} \begin{pmatrix}
(E_n + i m)I(n, p, x) \\
-i p_x I(n, p, x) \\
0
\end{pmatrix}, \\
\psi^2_p &= N_p e^{-i(E_p|t-p)} \begin{pmatrix}
0 \\
i p_x I(n-1, p, x) \\
-\sqrt{2}nB I(n, p, x)
\end{pmatrix}, \\
\psi^1_n &= N_n e^{i(E_n|t-p)} \begin{pmatrix}
\sqrt{2}nB I(n, p, x) \\
(E_n + i m)I(n-1, p, x) \\
0
\end{pmatrix}, \\
\psi^2_n &= N_n e^{i(E_n|t-p)} \begin{pmatrix}
0 \\
i p_x I(n-1, p, x) \\
\sqrt{2}nB I(n-1, p, x)
\end{pmatrix},
\end{align*}
\]

and the negative energy solutions as

\[
\begin{align*}
\psi^1_p &= N_p e^{-i(E_p|t-p)} \begin{pmatrix}
(E_n + i m)I(n-1, p, x) \\
-i p_x I(n-1, p, x) \\
0
\end{pmatrix}, \\
\psi^2_p &= N_p e^{-i(E_p|t-p)} \begin{pmatrix}
i p_x I(n-1, p, x) \\
(E_n + i m)I(n-1, p, x) \\
\sqrt{2}nB I(n-1, p, x)
\end{pmatrix},
\end{align*}
\]

where

\[
\begin{align*}
|E_n| &= \sqrt{2eBn + m^2 + p^2}, \\
N_n &= \frac{1}{\sqrt{2|E_n|(|E_n| + m)}}, \\
I(n, p, x) &= \left(\frac{eB}{\pi}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n \left[\sqrt{eB}(x - p/eB)\right] \times e^{-(eB/2)(x-p/eB)^2},
\end{align*}
\]

and \( I(n = -1, p, x) = 0 \). Moreover \( H_n(x) \) are Hermite polynomials, \( p \) is the fermion momentum in the \( y \) direction, \( n \) labels the Landau levels, and \( N_n \) is the normalization factor. We can readily make the following observations:

(i) In the LLL, \( \psi^1_N = \psi^2_N = 0 \) trivially. Thus LLL is nondegenerate whereas all other levels are degenerate in energy. This fact is independent of gauge and representation of \( \gamma^\mu \) [11].

(ii) Because of the appropriate choice of representation for the \( \gamma \) matrices, we do not need to solve the equation for \( (2 + 1) \) dimensions. Setting \( p_z = 0 \), we immediately get two decoupled sets of solutions corresponding to two species of fermions in a plane.

We now turn to the parity violating Dirac equation. In 4-dimensional representation of planar QED, we have at our disposal two matrices which commute with all three \( \gamma \) matrices entering the usual Dirac equation. We can take them to be \( \gamma^3 \) as defined before and \( \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \). It permits us to define two types of mass terms:

\[
\mathcal{L} = i\bar{\psi}(\not\partial - ieA)\psi - m_+\bar{\psi}\psi - m_0\bar{\psi}\tau\psi,
\]

where

\[
\tau = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

The additional mass term \( (m_\tau) \) in the Lagrangian violates parity though it preserves chiral symmetry, while the usual term \( m \) is invariant under parity transformations, but breaks chiral symmetry. The corresponding Dirac equation can be written as

\[
(i\gamma^\mu \partial_\mu + e\gamma^\mu A_\mu - m - m_\tau \tau)\psi = 0.
\]

We can tabulate the solutions for species \( A \) and \( B \) of the fermions as follows:

\[
\begin{align*}
\psi^A_p &= N^A_p e^{-i(E_p^A|t-p)} \begin{pmatrix}
(E_p^A + m_+)I(n, p, x) \\
-i p_x I(n-1, p, x) \\
0
\end{pmatrix}, \\
\psi^B_p &= N^B_p e^{-i(E_p^B|t-p)} \begin{pmatrix}
0 \\
i p_x I(n-1, p, x) \\
\sqrt{2}eB I(n-1, p, x)
\end{pmatrix}, \\
\psi^A_n &= N^A_n e^{i(E_n^A|t-p)} \begin{pmatrix}
\sqrt{2}eB I(n, p, x) \\
(E_n^A + m_+)I(n-1, p, x) \\
0
\end{pmatrix}, \\
\psi^B_n &= N^B_n e^{i(E_n^B|t-p)} \begin{pmatrix}
0 \\
i p_x I(n-1, p, x) \\
\sqrt{2}eB I(n-1, p, x)
\end{pmatrix},
\end{align*}
\]

where

\[
\begin{align*}
m_\pm &= m \pm m_\tau, \\
|E_p^A| &= \sqrt{2eBn + m^2}, \\
N^A_p &= \frac{1}{\sqrt{2|E_p^A|(|E_p^A| + m_+)}}, \\
|E_p^B| &= \sqrt{2eBn + m^2}, \\
N^B_p &= \frac{1}{\sqrt{2|E_p^B|(|E_p^B| + m_-)}},
\end{align*}
\]

Note that owing to the parity violation, the fermion spinning anticlockwise with respect to its direction of motion has mass \( m_+ \), whereas, the fermion spinning clockwise has mass \( m_- \). This difference of masses leads us to new features for the \( \psi\bar{\psi} \) condensate. Furthermore, when \( m_\tau = 0 \), we have \( m_\pm = m \), which implies \( |E_p^A| = |E_p^B| \) and \( N^A_p = N^B_p \), in such a fashion that the above solutions reduce to Eqs. (3) and (4) in the limit \( p_z = 0 \), provided we identify \( A \leftrightarrow 1 \) and \( B \leftrightarrow 2 \).
III. $\bar{\psi} \psi$ CONDENSATE

We first calculate the $\bar{\psi} \psi$ condensate in $(3 + 1)$ dimensions by expanding out the fermion field in its Fourier decomposition in terms of the complete basis of solutions provided by $\langle \psi_{p,N} \rangle$:

$$\psi(\vec{x}, t) = \sum_n \sum_{i} \int \frac{dp}{\sqrt{2\pi}} \int \frac{dp'}{\sqrt{2\pi}} [a_i(n, p, p_z) \psi_p^i + b_i^\dagger(n, p, p_z) \psi_p^i],$$

(7)

where $i = 1, 2$ except for the LLL. This is what the primed notation denotes. The sum over $n$ runs over all Landau levels. $p$ and $p_z$ are continuous momentum variables. The $a_i$ and $b_i$ are the particle and antiparticle destruction operators, respectively, obeying the anticommutation relations

$$\{a_i(n, p, p_z), a_j^\dagger(n', p', z') \} = \{b_i(n, p, p_z), b_j^\dagger(n', p', z') \} = \delta_{ij} \delta_{n n'} \delta(p - p') \delta(p_z - p'_z).$$

Evaluation of the condensate thus leads to

$$\langle \bar{\psi} \psi \rangle = -\frac{m}{(2\pi)^2} \sum_n \int_0^\infty dp [T^2(n - 1, p, x) + T^2(n, p, x)] \times \int_{-\infty}^\infty dp_z \frac{1}{\sqrt{m^2 + p_z^2 + 2n eB}}.$$

(8)

The integral and the sum over Landau levels in Eq. (8) are divergent and we need to introduce cutoffs for both. Moreover, we separate the LLL contribution from the rest of the Landau levels and employ the normalization conditions for the Hermite polynomials to arrive at

$$\langle \bar{\psi} \psi \rangle = -\frac{m eB}{2\pi^2} \ln \left[ \frac{1 + \sqrt{1 + X_n}}{\sqrt{X_0}} \right] - \frac{m eB n_{\text{LL}}}{\pi^2} \sum_{n=1}^\infty \ln \left[ \frac{1 + \sqrt{1 + X_n}}{\sqrt{X_0}} \right],$$

(9)

where $X_n = (m^2 + 2n eB)/\Lambda^2$. Through the method of Schwinger proper time, this condensate in terms of an integral over the proper time was obtained in [12]. As expected, $\langle \bar{\psi} \psi \rangle \to 0$ as $m \to 0$. Although the sum over the Landau levels diverges, the contribution of individual levels converges for growing $n$ as indicated by the fact that

$$n \to \infty \Rightarrow \ln \left( \frac{1 + \sqrt{1 + X_n}}{\sqrt{X_n}} \right) \to 0.$$ 

One can see that in the limit of $m$ very small, the leading contribution to the condensate comes from $n = 0$:

$$\langle \bar{\psi} \psi \rangle = -\frac{m eB}{4\pi^2} \ln \frac{\Lambda^2}{m^2}.$$ 

(10)

Thus in the LLL approximation, we recuperate the expression obtained in [9].

In $(2 + 1)$ dimensions, two types of mass terms ($m$ and $m_\tau$) are connected to two types of condensates $\langle \bar{\psi} \psi \rangle$ and $\langle \bar{\psi} \tau \psi \rangle$. We again resort to the second quantized representation of $\psi$ field.

$$\psi(\vec{x}, t) = \sum_n \sum_{i} \int \frac{dp}{\sqrt{2\pi}} [a_i(n, p) \psi_p^i + b_i^\dagger(n, p) \psi_p^i],$$

(11)

where $i = A, B$. This yields

$$\langle \bar{\psi} \psi \rangle = -\frac{eB}{2\pi} \sum_{n=1}^\infty \frac{m_+}{|E_n^A|} - \frac{eB}{2\pi} \sum_{n=1}^\infty \frac{m_-}{|E_n^B|}$$

(12)

$$\langle \bar{\psi} \tau \psi \rangle = \frac{eB}{2\pi} \sum_{n=1}^\infty \frac{m_+ - m_-}{|E_n^B|}.$$ 

Note that when $m = 0$, $\langle \bar{\psi} \psi \rangle \neq 0$ and thus chiral symmetry is broken by the interactions of electrons with the external magnetic field. Also, in the case $m_\tau = 0$, $\langle \bar{\psi} \tau \psi \rangle$ is not vanishing. This reflects that in QED3, these interactions lead to the violation of parity as well. Furthermore, in the latter case, the parity conserving condensate reduces to the result of Das [10]:

$$\langle \bar{\psi} \psi \rangle = -\frac{eB}{2\pi} \sum_{n=1}^\infty \frac{1}{|E_n^A|}.$$ 

(13)

At this point, it is convenient to notice that the projectors $(1 \pm \tau)/2$ allow us to entirely separate the A and B sectors of the theory [13]. This, in turn, permits us to write the following linear combinations of the condensates:

$$\langle \bar{\psi} \psi \rangle_+ = \langle \bar{\psi} \psi \rangle + \langle \bar{\psi} \tau \psi \rangle, \quad \langle \bar{\psi} \psi \rangle_- = \langle \bar{\psi} \psi \rangle - \langle \bar{\psi} \tau \psi \rangle.$$ 

Using the previous relations,

$$\langle \bar{\psi} \psi \rangle_+ = -\frac{eB}{2\pi} \sum_{n=1}^\infty \frac{m_+}{|E_n^A|},$$

(14)

$$\langle \bar{\psi} \psi \rangle_- = -\frac{eB}{2\pi} \sum_{n=1}^\infty \frac{m_-}{|E_n^B|}.$$ 

Consequently, $\langle \bar{\psi} \psi \rangle_+ \to 0$ as $m_+ \to 0$, a feature usual to condensates in $(3 + 1)$ dimensions. However, $\langle \bar{\psi} \psi \rangle_-$ retains a finite value as $m_- \to 0$. It is

$$\langle \bar{\psi} \psi \rangle_- = -\frac{eB}{2\pi} \text{sgn}(m_-).$$ 

(15)

This compares to the fact that massless theory exhibits a current of abnormal parity

$$\langle J^\mu(x) \rangle = \text{sgn}(m) \frac{e}{4\pi} \bar{F}^\mu(x),$$

(16)

where $^* F^\mu(x) = (1/2) e^{\mu \alpha \beta} F_{\alpha \beta}$ is the dual stress tensor, [14].

An important point to notice is the indeterminacy of the value of the condensates (12) and (15) when $m = m_\tau$. A similar feature was found in Ref. [15]. In that work, the
lack of differentiability of the effective potential of a model of QED3 with parity conserving and violating Yukawa mass terms was found along the line where both masses are equal.

Finally, expanding the condensates in the strong field limit, we obtain

\[
\langle \bar{\psi} \psi \rangle_- = -\frac{eB}{\pi} \frac{m_-}{2m_-} \left(\frac{3}{2}\right) + O \left(\frac{m_-^2}{eB}\right),
\]

\[
\langle \bar{\psi} \psi \rangle_+ = \frac{eB}{\pi} \frac{m_+}{2m_+} \left(\frac{3}{2}\right) + O \left(\frac{m_+^2}{eB}\right),
\]

(17)

The main results of the report are Eqs. (9), (12), (14), (15), and (17). These equations relate \( \bar{\psi} \psi \) condensate with the fermion mass and the magnetic field in \( 3 + 1 \) dimensions and \( 2 + 1 \) dimensions for parity conserving and violating Lagrangians. The result in \( 3 + 1 \) dimensions can have application in the physics of the early universe where the possible existence of magnetic/hypermagnetic fields has been shown to modify the behavior of the electroweak phase transition; see, for example, [16] and references therein. On the other hand, our findings for \( 2 + 1 \) dimensions are of relevance in parity conserving and violating effective theories describing high temperature superconductivity and the quantum Hall effect, where a magnetic field is an important ingredient of the dynamics. In particular, these could be of direct importance in the study of graphene, where in general, a kaleidoscopic variety of mass terms and order parameters can be defined which might manifest themselves in the experiment [17]. Work in this direction is in progress. Moreover, carrying out this work in the presence of a heat bath is likely to be of even greater interest in the study of the above-mentioned physical scenarios. All this is for the future.

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