Dynamical Masses and Confinement in QED$_3$

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Abstract. Dynamical Chiral Symmetry Breaking (DCSB) and Confinement are two crucial features of QCD which are responsible for the nature of the hadronic spectrum. A simpler model which exhibits both is quantum electrodynamics in (2+1) space-time dimensions, QED$_3$. A long standing debate in this model is the existence of a critical number of fermion families, $N_c$, above which DCSB ceases to take place. This was established from the solutions of the Schwinger-Dyson equations (SDEs), in the leading order of the $1/N$ expansion in the Landau gauge. Confinement has also been found to be absent in this scenario. In this work, we study the stability of the solutions to the said SDEs under a variation of gauge while still working with the bare vertex. We find that the Landau gauge is the only gauge which exhibits the above mentioned results. Away from this gauge, DCSB takes place for an arbitrarily large $N$ and confinement is reinstated. Attempting to understand this apparent inconsistency, we argue that in order to maintain the gauge covariance of the results, full vertex has to be employed in other gauges and/or constraints like the Landau-Khalatnikov-Fradkin transformations must be employed in going from Landau gauge to other gauges.

MOTIVATION

The SDEs form an infinite tower of relations among the Green’s functions of a given quantum field theory. Due to its non-Abelian structure, the study of the corresponding SDEs of QCD become a hard nut to crack. A simpler model which also exhibits DCSB and Confinement [1] is quantum electrodynamics in (2+1)-dimensions, QED$_3$. Besides being a toy model for QCD, QED$_3$ is an interesting theory in its own. It has important applications in condensed matter physics, e.g., in high temperature superconductors [2], quantum Hall effect, and more recently, Graphene [3].

In the field of DCSB, a long standing controversy is the existence of a critical behavior of QED$_3$. Appelquist et al. [4] observed that the solutions to the gap equation are sensitive to the number of massless fermion families $N$ considered in the model, in such a fashion that above a critical value $N_c = 32/\pi^2$, only the chiral symmetry preserving solution exists. These results were obtained in the Landau gauge truncating the SDEs at the leading order in the $1/N$ expansion, which amounts to consider the bare vertex and neglect effects of the wavefunction renormalization in the gap equation. These results were refuted by Pennington and co-workers [5], who criticized the use of a perturbation theory inspired truncation in the study of a non-perturbative phenomenon. They noticed that taking into account wavefunction renormalization effects and considering vertex corrections, chirally asymmetric solutions exist for arbitrarily large values of $N$, but the amount of dynamically generated mass is exponentially suppressed as $N$ grows. Adding to the controversy, lattice simulations seemed to favor Appelquist results [6], but due to the complications of collocating small masses on the lattice, none of these scenarios can
still be ruled out [7]. Even accepting that $N_c$ is finite, its precise value still depends on the details of the study, and has been found in the range 3-6 [8].

In this contribution, attempting to shed light into the discussion, we test the covariance of the solutions found by Appelquist et al. [4]. In the next section, we review the scenario of criticality solving the gap equation at the leading order of the $1/N$ expansion in covariant gauges. In section 3 we perform a confinement test on these solutions. From these results, Landau gauge seems to be an isolated gauge in the sense that it is the only one in which $N_c$ is finite and confinement is absent. We then argue that in order to maintain the covariance of the truncation under variations of gauge, the full vertex and/or identities like the Landau-Khalatnikov-Fradkin transformation (LKFT) should be employed. This is the subject of section 4, which precedes our concluding section.

**DYNAMICAL MASSES**

A common starting point to study DCSB is the gap equation, which in QED$_3$ is

$$S^{-1}(p) = S_0^{-1}(p) - ie^2 \int \frac{d^3k}{(2\pi)^3} \gamma^\mu S(k) \Gamma^\nu(k, p) \Delta_{\mu\nu}(q), \quad (1)$$

where $\Delta_{\mu\nu}$ is the photon propagator, $\Gamma^\nu(k, p)$ is the fermion-photon vertex, $e^2$ is the electromagnetic coupling and $q = k - p$. The most general form of the fermion propagator is $S(p) = F(p)/(\not{p} - M(p))$, where $F(p)$ is referred to as the wavefunction renormalization and $M(p)$ is the mass function. Its bare counterpart is $S_0(p) = 1/(\not{p} - m)$ with $m$ the fermion bare mass. The photon propagator and the fermion boson vertex in this expression are coupled to the rest of the infinite tower of SDEs. Following Appelquist et al. [4], we consider $N$ massless fermions and truncate this tower in the $1/N$ expansion. A suitable expansion parameter in this case is $\alpha = e^2 N$, which is kept small as $N \to \infty$. At the leading order, we choose $\Gamma^\nu(k, p) = \gamma^\nu$ and neglect wavefunction renormalization effects, $F(p) = 1$. The photon propagator is

$$\Delta_{\mu\nu} = -\frac{1}{q^2 - \alpha q/8} \left( g_{\mu\nu} - \frac{q\mu q\nu}{q^2} \right) + \xi \frac{q\mu q\nu}{q^4}, \quad (2)$$

where $\xi$ is the covariant gauge parameter. With all the above, the gap equation becomes

$$M(p) = \frac{\alpha}{4\pi^2 N p} \int_0^\infty dk \frac{k M(k)}{k^2 + M^2(k)} \left[ \ln \left| \frac{\alpha/8 + k + p}{\alpha/8 + |k - p|} \right| + \xi \ln \left| \frac{k + p}{k - p} \right| \right], \quad (3)$$

result obtained after a Wick rotation and integration of angular variables. We solve this equation as a function of $N$ for various values of $\xi$ and study how much dynamical mass, which can be quantified by $M(0)$, is generated varying $N$. Results are displayed in Fig. 1.

We observe that in the Landau gauge, $\xi = 0$, for $N = N_c \simeq 3.2$ DCSB ceases to take place, in agreement with Appelquist et al. [4]. However, for any $\xi \neq 0$, DCSB is found for arbitrarily large values of $N$, hence rendering the Landau gauge isolated. Below we perform a confinement test to the solution of the gap equation (3).
FIGURE 1. Dynamical mass as a function of $N$ in different covariant gauges.

CONFINEMENT

Confinement, that is, the impossibility of color excitations to propagate to a detector, can be tested through $n$-point Euclidean (Schwinger) functions by means of the violation of the axiom of reflection positivity. This axiom states that if a propagator is associated to a physical, observable state, then

$$D(t) = \int d^3x \int \frac{d^3p}{(2\pi)^3} e^{i(tp_0 + x \cdot \vec{p})} \frac{F(p)M(p)}{p^2 + M^2(p)} \geq 0. \quad (4)$$

For instance, if there is a stable asymptotic state of mass $m$ described by the propagator, then $\Delta(t) \sim e^{-mt}$ as $t \to \infty$. However, if the propagator supports two complex mass-like singularities $\mu = a \pm ib$, then $\Delta(t) \sim e^{-at} \cos(bt + \delta)$ in this limit. Performing a confinement test in the form advocated in [9, 10] to the solutions found in the previous section, we obtain the results shown in Fig. 2. These curves can be fitted according to $\Delta(t) = Ae^{-at} \cos[\xi(bt + \delta)]$, which shows that the oscillatory behavior of the Schwinger function, i.e., confinement, can be found in all but the Ladau gauge. Thus, this gauge is also isolated regarding the scenario of confinement. This apparent inconsistency arises from a naive variation of gauge. The manner to retain the covariance of the gap equation and the physical observables associated with it is discussed in the following section.

GAUGE COVARIANCE

Physical observables are independent of the choice of gauge. In our case, this means that $N_c$ should be the same in all covariant gauges. Nevertheless, our findings in Sect. 2 seem
to indicate that only in Landau gauge $N_c$ is finite. The problem arises since our truncation scheme does not preserve the covariance properties of the gap equation: Although the bare vertex and the assumption that $F(p) = 1$ seem to be enough to satisfy gauge identities like the Ward-Takahashi identity (WTI) [10], this is not necessarily the case in other gauges. A new vertex should be chosen and effects of $F(p)$ should be incorporated each time we solve the gap equation for a new $\xi \neq 0$ to maintain the covariance of the results. Such a formidable task can be accomplished if the full fermion-boson vertex is employed in the truncation and/or owing to the gauge covariance properties of Green functions, which in QED are enclosed in the so-called Landau-Khalatnikov-Fradkin transformations (LKFT) [11]. These transformations preserve the covariance of WTI and SDEs, and lead to gauge independent physical observables, as advocated in [10]. LKFT in QED$_3$ establish that, starting from the Landau gauge solutions $F_0(p)$ and $M_0(p)$, in other gauge, the propagator can be obtained through

$$
\frac{F_\xi(p)}{p^2 + M_\xi^2(p)} = \frac{a}{\pi p^2} \int_0^\infty dk \frac{k^2 F_0(k)}{k^2 + M_0^2(k)} \left[ \frac{1}{\lambda^-} + \frac{1}{\lambda^+} + \frac{1}{2kp} \ln \left| \frac{\lambda^-}{\lambda^+} \right| \right]
$$

$$
\frac{F_\xi(p) M_\xi(p)}{p^2 + M_\xi^2(p)} = \frac{a}{\pi p} \int_0^\infty dk \frac{k F_0(k) M_0(k)}{k^2 + M_0^2(k)} \left[ \frac{1}{\lambda^-} - \frac{1}{\lambda^+} \right],
$$

where $\lambda^{\pm} = a^2 + (k \pm p)^2$ and $a = e^2 \xi/(8\pi)$. The above relations make it obvious that wavefunction renormalization effects cannot be simply neglected in other gauges. Furthermore, the second of these ensures that if $M_0(p) = 0$, then $M_\xi(p) = 0$ in other gauges as well. As a consequence, in our case, since $M_0(p) = 0$ at $N_c$, this value should be the same in all other gauges. Thus, we confirm that $N_c$ is indeed a physical observable, whose value remains the same, irrespective of the gauge we choose to work in.
On the other hand, QED$^3$ is known to be a confining theory [1]. This can be seen from the asymptotic form of the classical potential

$$V(r) \overset{r \to \infty}{=} \frac{e^2}{2\pi} \frac{1}{1 + \Pi(0)} \ln(e^2 r) + \text{const.} + h(r), \quad (6)$$

where $\Pi(0)$ is the value of the vacuum polarization at the origin and $h(r)$ is a function which falls-off at least as $1/r$ [9]. In the quenched version of QED$^3$, $\Pi(0) = 0$ and thus $V(r)$ becomes logarithmically confining, i.e., an infinite amount of energy is required to bring two static charges apart. In our case, since we are considering massless fermions ($\Pi(q) = \alpha/(8q)$ in Landau gauge), which cost no energy to produce and can travel to infinity screening the interaction completely, confinement is washed away. This confirms our findings in Sect. 3 in Landau gauge. In other gauges, the loss of covariance induces a spurious gauge dependent mass to virtual fermions, thus rendering $\Pi(0)$ finite and re-instating confinement. LKFT, however, ensures that such an spurious mass is absent, and thus that confinement is also absent in any other gauge. In this fashion, all predictions of SDEs are consistent with the gauge covariance of QED$^3$.

**FINAL REMARKS**

In gauge theories, the only systematic way of truncating SDEs and preserve the gauge symmetry is perturbation theory. At the non-perturbative level, care, however, must be taken for this to be acomplished. A careless truncation induces spurious gauge dependence to physical observables, leading to meaningless predictions in some cases. In QED4, this issue has been studied extensively in connection with providing a complete interaction vertex, [12]. In QED3, a corresponding start has also been made, [13]. In this contribution, we have solved in covariant gauges the gap equation of QED3 with $N$ families of massless fermions truncated at the leading order of the $1/N$ expansion in the study of DCSB and confinement. We found that Landau gauge appears to be an isolated gauge in the sense that it is the only one in which DCSB ceases to take place if $N$ exceeds the critical value $N_c = \pi^2/32$ and confinement is absent. In other gauges, chirally asymmetric, confining solutions can be found for arbitrarily large values of $N$. These apparently unphysical conclusions arose from the lack of covariance of the gap equation. Have the full fermion-boson vertex been employed in the truncation, results should not be gauge dependent, but finding its complete expresion is a formidable task. The LK transformations, however, ensure that $N_c$ is a gauge independent quantity, and confinement is absent in all covariant gauges. A more realistic truncation of SDEs would consist in a model which considers feedback of the interaction between fermions and photons into their respective propagators. Only then one can definitively confirm or discard the existence of a finite value of $N_c$ and thus of the chiral-symmetry-restauration/deconfinement phase transition. This work is in progress.

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